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# A Relation-algebraic Approach to Simple Games ⋆

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**Abstract.** Simple games are a powerful tool to analyze decision-making and coalition formation in social and political life. In this paper, we present relation-algebraic models of simple games and develop relational algorithms for solving some basic problems of them. In particular, we test certain fundamental properties of simple games (being monotone, proper, respectively strong) and compute specific players (dummies, dictators, vetoers, null players) and coalitions (minimal winning coalitions and vulnerable winning coalitions). We also apply relation-algebra to determine central and dominant players, swingers and power indices (the Banzhaf, Holler-Packel and Deegan-Packel indices). This leads to relation-algebraic specifications, which can be executed with the help of the BDD-based tool RelView after a simple translation into the tool's programming language. In order to demonstrate the visualization facilities of RelView we consider an example of the Catalonian Parliament after the 2003 election.

**JEL Classification**: C71, C88, C63, C65, D72

**Keywords**: relation algebra, RelView, simple game, winning coalition, swinger, dominant player, central player, power index

## 1 Introduction

A *simple game* is a cooperative game in which only two types of coalitions can be formed, to wit, *winning* coalitions and *losing* ones. A winning coalition takes it all while a losing coalition receives nothing. Since winning seems to be the essence of politics, simple games are extremely suitable for analyzing political situations. A particular case of a winning coalition is a *minimal winning* coalition in which every member is needed to make it winning. In other words, the removal of any of its members will make it a losing coalition. A particular case of a losing coalition is a *blocking* coalition whose complement is also losing. Although a blocking coalition cannot enforce a decision, it can prevent the enforcement of any decision.

Important concepts in the theory of simple games are the concepts of swinger, veto-player, dictator, and dummy. A *swinger* of a winning coalition is a member of the coalition whose removal makes the coalition losing. A *veto-player* is a player who is in every minimal winning coalition. Under the monotonicity assumption, a veto-player must be in every winning coalition, and hence no coalition can win without a veto-player. A simple game is *weak* if it contains a veto-player. If one player forms the only minimal winning coalition, then this player is called a *dictator*. There is obviously an essential difference between a veto-player and a dictator. A

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dictator can win on his own, he can enforce any decision without help of the other players. In contrast, a veto-player is needed to win, but cannot win on his own. He only can block the decision-making process, not enforce a decision. A *dummy* is a player who is a member of no minimal winning coalition. A dummy is a powerless player. It cannot make a winning coalition losing or a losing coalition winning.

Apart from defining simple games as those in which each coalition either wins or loses, see, e.g., von Neumann and Morgenstern (1944, [28]), Shapley and Shubik (1954, [42]), Peleg and Sudhölter (2003, [32]), some scholars considered combining simple games into one game or decomposing a game into components. O'Neill and Peleg (2008, [29]), for instance, study several types of composition rules (e.g., the lexicographic, projection, veto, and product rules) to combine two proper games into one proper game, where the success of a coalition depends on whether it wins, blocks, or loses in each component.

When studying coalition formation, one of the most important issues is to identify some key players in the game. Several concepts of such specific players have been developed in the literature. In Peleg (1981, [31]) a theory of coalition formation in simple games with dominant players has been developed. Roughly speaking, a *dominant player* in a simple game is a player who holds a strict majority within a winning coalition; see also Peleg (1981, [31]). The dominant player is a 'policy blind' or 'office seeking' concept. The dominant players are the most powerful players of a game. Such players neither must exist nor must be unique. However, Peleg proved that in weak simple games and weighted majority games at most one dominant player may occur. Games with dominant players are called *dominated*. For further discussions, see, e.g., Peleg (1981, [31]), van Deemen (1989, [12]) and van Roozendaal (1990, [34]).

Another interesting concept is the concept of a *central player* introduced in Einy (1985, [16]). In contrast to the dominant player, the central player is a 'policy oriented' or 'policy seeking' concept. It is not hard to prove that there exists at most one central player in a simple game. In order to find the central player of a game, the players must be ordered on a relevant policy dimension, and the particular position of the central player makes him very powerful. A simple game in which a central player exists is called *centralized*. An empirical analysis of the importance and effect of dominant and central parties on cabinets in Western multiparty democracies has been examined e.g., in van Roozendaal (1992a, 1992b, 1993, 1997, [35, 36, 37, 38]). In van Roozendaal (1997, [38]), for instance, the author argues that there are certain theoretical reasons by which governments including dominant parties should be more stable than governments without dominant parties. Moreover, van Roozendaal shows, by analyzing government survival in 12 countries between 1945 and 1989, that such an effect indeed exists in real politics. Furthermore, in van Roozendaal (1993, [37]), the importance of the central player in Dutch cabinets has been shown.

Relatively simple examples of problems in the field of simple games are frequently already too complicated to be solved by hand. Therefore, it is useful to have computer programs available to deal with such problems. But because many problems appearing with simple games are intractable in terms of complexity theory, for instance, #P-complete or NP-hard, even a computer program may not be able to deal with somewhat larger examples.

One usually does not immediately think of computer programs based on relation-algebraic formulations of the concepts in question. However, since some decades relation algebra is used successfully for formal problem specification, prototyping, and algorithm development. See e.g., Schmidt and Ströhlein (1993, [40]), Brink et al. (1997,[10]) and de Swart el al. (2003, [43]).

Relations are well suited for modeling and reasoning about many discrete structures (like graphs, hypergraphs, Petri nets, orders, lattices) and, due to the easy mechanization (using, for instance, Boolean matrices) also for computations on them.

RelView is a visual computer system for the visualization and manipulation of relations and for relational prototyping and programming. The tool is written in the C programming language and makes full use of the X-windows graphical user interface. Among the advantages of RelView are, for instance, short and concise programs which frequently consist of only a few lines expressing the relation-algebraic expression of the notions in question. For details and applications, see e.g., Berghammer et al. (1996, [5]), Behnke et al. (1998, [3]), Berghammer et al. (2003, [9]), Berghammer and Milanese (2006, [7]), Berghammer and Fronk (2006, [4]) and Berghammer et al. (2009, [8]). *One of the aims of this paper is to apply the relation-algebraic approach to the key concepts defined in the domain of simple games.* Taking into account that all these concepts are important both from a theoretical and an empirical point of view, the application of the relation-algebraic approach, on the one hand being a mathematical formal approach and on the other hand giving an immediate access to the RelView implementation, is very useful. Because the RelView tool has a very efficient BDD (Binary Decision Diagram) implementation of relations, developed in the course of the Ph.D. theses Leoniuk (2001, [25]) and Milanese (2003, [27]) and shortly described in Berghammer et al. (2002, [6]), it is able to deal with non-trivial simple games that appear, e.g., in practical political life. In addition, this tool allows animations and has visualization facilities in the form of matrices and graphs, which are not easily found in other software tools and which are most helpful for fully comprehending difficult concepts and for understanding and testing the programs.

One of the most important elements of simple games is to measure power of players. To this end, during the last decades some so-called power indices have been proposed, e.g., the Shapley-Shubik index by Shapley and Shubik (1954, [42]), the Banzhaf index by Banzhaf (1965, [2]), the Deegan-Packel index by Deegan and Packel (1978, [11]), the Johnston index by (Johnston, 1978, [21]) and the Holler-Packel index by Holler (1982, [19]) and Holler and Packel (1983, [20]), They are based on different models for power and, therefore, their use and informative value depends on the context in which they are applied. Axiomatic characterizations, as, e.g., presented in Dubey (1975, [13]), Dubey and Shapley (1979, [15]), Lehrer (1988, [24]), Laruelle and Valenciano (2001, [23]), Dubey et al. (2005, [14]), Lorenzo-Freire et al. (2007, [26]) and Alonso-Meijide et al. (2008, [1]) are helpful for the appraisal of their applicability. For an extensive analysis of most of the power indices see, e.g., Owen (1995, [30]) and Felsenthal and Machover (1998, [17]).

Although power indices have been studied in detail in the voting power literature, usually the standard approach has been applied to analyze them. It is useful, however, to apply other (interdisciplinary) approaches to power indices. One such very interesting application is presented in Saari and Sieberg (2000, [39]), where the authors use linear algebra and geometry to show and explain some surprising properties of power indices related to different rankings obtained by different power indices. *In this article we present an application of another field of mathematics (viz. relation algebra) to power indices.* Measuring power is of importance and can be applied to all kinds of organizations, for example, to political bodies, international economic organizations, and to business settings. Hence, our relation-algebraic approach to voting power is particularly useful, because it allows to apply the very efficient RelView tool to compute power indices.

The remainder of the paper is structured as follows. In Section 2 the main game-theoretic concepts that we deal with in the paper are presented. In Section 3 we start with a brief introduction to relation algebra and present a few relation algebraic constructions which are used later in the paper. The core of the paper is Section 4. Here we start in the first subsection with two relation algebraic models of simple games, show how they can be transformed into each other and we give relation-algebraic characterizations of monotone simple games, of proper and of strong simple games. Thereby, the visualization facilities of RelView are demonstrated in the case of the parliament of Catalonia after the 2003 election. In the next subsection we give the relation-algebraic expressions for the set of minimal winning coalitions, of the notion of being a swinger in a given coalition and of the notion of vulnerable winning coalition. We also give relation-algebraic formulations for the sets of dummies, vetoers, dictators and null players, respectively. Again, these notions are illustrated in the case of the Catalonian parliament by running RelView programs based on the respective algebraic expressions. The third subsection is devoted to the development of relation-algebraic expressions for the set of central players and for notions around dominant players. The RelView tool enables us to show the Hasse-diagram of the more-desirable relation between coalitions in the parliament of Catalonia after the 2003 election. In the final subsection we define the Banzhaf, the Holler-Packel and the Deegan-Packel power indices in terms of relation algebra and again demonstrate the ability of the RelView tool to compute these indices for the Catalonian parliament. Some concluding remarks are presented in Section 5.

## 2 Game-theoretic Preliminaries

In this section we present some basic concepts of the theory of simple games that we refer to in the paper. More extensive treatments of simple games can be found, for instance, in Shapley (1962, [41]), Owen (1995, [30]), Felsenthal and Machover (1998, [17]) and Peleg and Sudhölter (2003, [32]).

### 2.1 Simple Games

Following Peleg and Sudhölter (2003, [32]), a simple game is a pair $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $\mathcal{W}$ is a subset of the powerset $2^N$. Any element of $2^N$ is called a *coalition*. A coalition $S$ with $S \in \mathcal{W}$ is called *winning*, while those with $S \notin \mathcal{W}$ are called *losing*. A simple game $(N, \mathcal{W})$ is called *monotone* if $\mathcal{W}$ is an up-set in the order $(2^N, \subseteq)$, i.e., for all $S, T \in 2^N$ from $S \in \mathcal{W}$ and $S \subseteq T$ it follows $T \in \mathcal{W}$. A *voting game* is a monotone simple game $(N, \mathcal{W})$ with $\mathcal{W} \neq \emptyset$ and $\emptyset \notin \mathcal{W}$. The latter two axioms exclude trivial games. A simple game is *proper* if the complement of a winning coalition is always losing, and *strong* if the complement of any losing coalition is winning. A simple game is *decisive* if it is both proper and strong. In the context of voting games, for instance, being proper is interesting since it is equivalent to the fact that any pair of winning coalitions has a player in common and being strong is interesting since here no *blocking coalitions* can occur, i.e., coalitions $S$ such that $S$ and $\overline{S}$ are losing.

An important class of games are *weighted majority games*. They are omnipresent, in particular, if groups (commissions, boards, ...) have to come to decisions and the members have unequal power. Usually, a weighted majority game with $n$ players is represented by a $n+1$-tuple

$$[q; w_1, w_2, ..., w_n], \tag{1}$$

where $q \in \mathbb{N}$ denotes the quota needed for a coalition to win, and $w_k \in \mathbb{N}$ is the weight assigned to player $k \in N$. By $w(S) = \sum_{k \in S} w_k$ we define the *weight* of a coalition $S$. A coalition $S$ is then winning if its weight is at least as large as $q$, that is, $S \in \mathcal{W}$ if and only if $w(S) \geq q$.

### 2.2 Minimal Winning Coalitions and Related Notions

Von Neumann and Morgenstern (1944, [28]) introduced the concept of a *minimal winning coalition* of a simple game $(N, \mathcal{W})$, that is a coalition $S \in \mathcal{W}$ such that $T \notin \mathcal{W}$ for all coalitions $T \subset S$. Less restrictive is the notion of a *vulnerable winning coalition* $S$. Here, besides $S \in \mathcal{W}$, it is demanded that there exists a player $k \in S$ such that $T \notin \mathcal{W}$ for all $T \subseteq S \setminus \{k\}$. In case of a monotone game the latter property is equivalent to the existence of $k \in S$ such that $S \setminus \{k\} \notin \mathcal{W}$. Such a player $k \in S$ is called a *swinger* (or *critical player*) of $S$. These concepts are, e.g., of importance when measuring the *power* of players.

Apart from swingers, one can distinguish other specific players in a simple game, depending on their relation to minimal winning coalitions. Let $(N, \mathcal{W})$ be a simple game and $k \in N$. Then $k$ is called a *dummy* if it does not belong to a minimal winning coalition, a *vetoer* if it is a member of each minimal winning coalition, and a *dictator* if $\{k\}$ is the only minimal winning coalition. Finally, $k$ is a *null player* if for each coalition $S \in 2^N$ it holds $S \cup \{k\} \in \mathcal{W}$ if and only if $S \in \mathcal{W}$.

### 2.3 Central and Dominant Players

As already mentioned in the introduction, the concept of a *central player* has been introduced in Einy (1985, [16]). Here it is assumed that the players of the game under consideration are ordered with respect to their policy positions. In political science one usually uses a left-to-right spectrum and the most important case is that the parties are ordered according to their stands in social and economic matters.

Given a simple game $(N, \mathcal{W})$ and a *policy order* of the players in the form of a linear strict order relation $P : N \leftrightarrow N$, i.e., a relation for which $P \subseteq \bar{\mathsf{I}}$ (irreflexivity), $PP \subseteq P$ (transitivity) and $P \cup P^{\mathsf{T}} = \bar{\mathsf{I}}$ (linearity) hold, player $k \in N$ is said to be *central* if the connected coalition $\{j \in N : P_{j,k}\}$ to the 'left' of $k$ as well as the connected coalition $\{j \in N : P_{k,j}\}$ to the 'right' of $k$ are not winning, but both can be turned into winning coalitions when $k$ joins them.

Based on two desirability-relationships between coalitions, in Perleg (1981, [31]) the concept of dominance and *dominant players* is developed. Let $(N, \mathcal{W})$ be a simple game, $S, T \in 2^N$ be coalitions and $k \in N$ be a player. Then $S$ is called *as least as desirable* as $T$, written as $S \geq_D T$, if for all $U \in 2^N$ from $U \cap S = \emptyset$, $U \cap T = \emptyset$ and $U \cup T \in \mathcal{W}$ it follows $U \cup S \in \mathcal{W}$. $S$ is said to be *more desirable* than $T$, written as $S >_D T$, if $S \geq_D T$ but not $T \geq_D S$. Finally, $k$ *dominates* $S$, written as $k \gg S$, if $k \in S$ and $\{k\} >_D S \setminus \{k\}$, and $k$ is *dominant* if there exists a $S \in \mathcal{W}$ such that $k \gg S$. If $k$ dominates $S$, then $k$ can form a winning coalition with players outside of $S$ while $S \setminus \{k\}$ is not able to do this. The dominant players are the most powerful players of the game. Such players neither must exist nor must be unique. However, Peleg proved that in weak simple games and weighted majority games at most one dominant player may occur. Games with dominant players are called dominated.

### 2.4 Power Indices

In this section we recapitulate power indices that we deal with in the paper. One of the main power indices that can be found in the literature is the *Banzhaf index* (Banzhaf, 1965, [2]). Let a monotone simple game $(N, \mathcal{W})$ and a player $k \in N$ be given. Then the absolute Banzhaf index $\overline{B}\,(k)$ of $k$ and the normalized Banzhaf index $B(k)$ of $k$ are defined as follows, where $n$ is the number of players:

$$\overline{B}(k) := \frac{|\{S \in \mathcal{W} \mid k \text{ swinger of } S\}|}{2^{n-1}} \qquad\qquad B(k) := \frac{\overline{B}(k)}{\sum_{j \in N} \overline{B}(j)} \tag{2}$$

Another well-known power index that we study in the paper is the *Holler-Packel index* of Holler (1982, [19]) and Holler and Packel (1983, [20]). Since a minimal winning coalition $S$ coincides with the set of its swingers, the absolute Holler-Packel index $\overline{H}(k)$ of $k$ can be specified in a way very similar to the definition of $\overline{B}(k)$ in (2). Compared with the definition of $\overline{B}(k)$, only in the numerator the set $\mathcal{W}$ is to be replaced by the set $\mathcal{W}_{\min}$ of minimal winning coalitions and the denominator is to be changed to $|\mathcal{W}_{\min}|$. The definition of the normalized Holler-Packel index exactly corresponds to the definition of $B$ via $\overline{B}$ in (2). Hence, we have:

$$\overline{H}(k) := \frac{|\{S \in \mathcal{W}_{\min} \mid k \text{ swinger of } S\}|}{|\mathcal{W}_{\min}|} \qquad\qquad H(k) := \frac{\overline{H}(k)}{\sum_{j \in N} \overline{H}(j)} \tag{3}$$

A power index that is related to minimal winning coalitions is also the *Deegan-Packel index* of Deegan and Packel (1978, [11]). Given a monotone simple game $(N, \mathcal{W})$ with set $\mathcal{W}_{\min}$ of minimal winning coalitions, the Deegan-Packel index $D(k)$ assigns to each player $k \in N$ the following number, where the set $\mathcal{W}_{\min}^{(k)}$ denotes the set of all minimal winning coalitions of the game which contain $k$:

$$D(k) := \frac{1}{|\mathcal{W}_{\min}|} \sum_{S \in \mathcal{W}_{\min}^{(k)}} \frac{1}{|S|} \tag{4}$$

All the concepts introduced in the literature of simple games and recapitulated in this section will be defined again in Section 4 in terms of relation algebra. Before it will be done, in the following section we present the relation-algebraic notions that we need for the realization of this task.

## 3 Relation-algebraic Preliminaries

In this section, we first recall the basics of relation algebra. Next, we introduce some further relation constructions which are used in the remainder of the paper. For more details on relations and relation algebra, see, e.g., Schmidt and Ströhlein (1993, [40]) or Brink et al. (1997, [10]).

### 3.1 Relation Algebra

If $X$ and $Y$ are sets, then a subset $R$ of the Cartesian product $X \times Y$ is called a (binary) relation with *domain* $X$ and *range* $Y$. We denote the set (in this context also called type) of all relations with domain $X$ and range $Y$ by $[X \leftrightarrow Y]$ and write $R : X \leftrightarrow Y$ instead of $R \in [X \leftrightarrow Y]$. If $X$ and

$Y$ are finite sets of size $m$ and $n$, respectively, then we may consider a relation $R : X \leftrightarrow Y$ as a Boolean matrix with $m$ rows and $n$ columns. The Boolean matrix interpretation of relations is well suited for many purposes and also used as one of the graphical representations of relations within the RelView tool. Therefore, in this paper we often use Boolean matrix terminology and notation. In particular, we speak of rows, columns and entries of relations and write $R_{x,y}$ instead of $\langle x, y\rangle \in R$ or $x\,R\,y$ to express that $x$ and $y$ are related via $R$.

In the present paper we use the following basic operations of relation algebra: $R^{\mathsf{T}}$ (*transposition*), $\overline{R}$ (*complement*), $R \cup S$ (*union*), $R \cap S$ (*intersection*) and $RS$ (*composition*). As special relations we use $\mathsf{O}$ (*empty relation*), $\mathsf{L}$ (*universal relation*), and $\mathsf{I}$ (*identity relation*). If $R$ is included in $S$ we write $R \subseteq S$ and equality of $R$ and $S$ is denoted as $R = S$. We assume the reader to be familiar with the component-wise descriptions of these notions, e.g., that, given $R : X \leftrightarrow Y$, $x \in X$ and $y \in Y$, it holds $R^{\mathsf{T}}_{x,y}$ if and only if $R_{y,x}$ and $\overline{R}_{x,y}$ if and only if $\neg R_{y,x}$.

### 3.2 Modelling of Sets

Relation algebra offers some simple and elegant ways to model subsets of a given set or, equivalently, predicates on this set. In this paper we will use vectors, is-element relations and injective mappings for this task.

A *vector* $v$ is a relation $v$ with $v = v\mathsf{L}$. As for a vector, therefore, the range is irrelevant, we consider in the following mostly vectors $v : X \leftrightarrow \mathbf{1}$ with a specific singleton set $\mathbf{1} := \{\perp\}$ as range and omit in such cases the second subscript, i.e., write $v_x$ instead of $v_{x,\perp}$. Analogously to linear algebra we will use lower-case letters to denote vectors. A vector $v : X \leftrightarrow \mathbf{1}$ can be considered as a Boolean matrix with exactly one column, i.e., as a Boolean column vector, and *represents* (or: is a representation of) the subset $\{x \in X \mid v_x\}$ of $X$. A non-empty vector $v$ is a *point* if $vv^{\mathsf{T}} \subseteq \mathsf{I}$, i.e., it is *injective*. This means that it represents a singleton subset of its domain or an element from it if we identify a set $\{x\}$ with the element $x$. In the matrix model, hence, a point $v : X \leftrightarrow \mathbf{1}$ is a Boolean column vector in which exactly one entry is 1.

Given $y \in Y$, with $R^{(y)}$ we denote the $y$-column of the relation $R : X \leftrightarrow Y$. That is, $R^{(y)}$ has type $[X \leftrightarrow \mathbf{1}]$ and for all $x \in X$ are $R^{(y)}_x$ and $R_{x,y}$ equivalent. To compare the columns of two relations $R$ and $S$ with the same domain $X$ and possible different ranges $Y$ and $Y'$, we use the symmetric quotient

$$\mathrm{syq}(R, S) := \overline{R^{\mathsf{T}}\,\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}} S} \tag{5}$$

of them. The type of $\mathrm{syq}(R, S)$ is $[Y \leftrightarrow Y']$, and transforming (5) into a component-wise notation we have for all $y \in Y$ and $y' \in Y'$ that $\mathrm{syq}(R, S)_{y,y'}$ if and only if $R^{(y)} = S^{(y')}$, i.e., if and only if for all $x \in X$ the relationships $R_{x,y}$ and $S_{x,y'}$ are equivalent.

As a second way to deal with sets we will apply the relation-level equivalents of the set-theoretic symbol $\in$, that is, *is-element relations* $\mathsf{E} : X \leftrightarrow 2^X$ between $X$ and its powerset $2^X$. These specific relations are defined by demanding for all elements $x \in X$ and sets $Y \in 2^X$ that $\mathsf{E}_{x,Y}$ if and only if $x \in Y$. A simple Boolean matrix implementation of is-element relations requires an exponential number of bits. However, in Leoniuk (2001, [25]) an ingenious implementation of $\mathsf{E} : X \leftrightarrow 2^X$ using reduced ordered binary decision diagrams (ROBDDs) is developed, where the number of BDD-vertices is linear in the size of the base set $X$. This implementation is part of RelView.

Finally, we will use injective mappings for modeling sets. Given an injective function $\imath : Y \to X$ in the usual mathematical sense, we may consider $Y$ as a subset of $X$ by identifying it with its image under $\imath$. If $Y$ is actually a subset of $X$ and $\imath$ is given as a relation of type $[Y \leftrightarrow X]$ such that $\imath_{y,x}$ if and only if $y = x$ for all $y \in Y$ and $x \in X$, then the vector $\imath^{\mathsf{T}}\mathsf{L} : X \leftrightarrow \mathbf{1}$ represents $Y$ as a subset of $X$ in the sense above. Clearly, the transition in the other direction is also possible, i.e., the generation of a relation $\mathrm{inj}(v) : Y \leftrightarrow X$ from the vector representation $v : X \leftrightarrow \mathbf{1}$ of the subset $Y$ of $X$ such that for all $y \in Y$ and $x \in X$ we have $\mathrm{inj}(v)_{y,x}$ if and only if $y = x$. We obtain $\mathrm{inj}(v)$ by removing from $\mathsf{I} : X \leftrightarrow X$ all rows which correspond to a 0-entry in $v$. The relation $\mathrm{inj}(v)$ is an injective mapping in the relation-algebraic sense; see, e.g., Section 4.2 of Schmidt and Ströhlein (1993, [40]). A combination of such relations with is-element relations allows a *column-wise representation* of sets of subsets. More specifically, if the vector $v : 2^X \leftrightarrow \mathbf{1}$ represents a subset $\mathcal{S}$ of $2^X$ in the sense above, i.e., $\mathcal{S}$ equals the set $\{Y \in 2^X \mid v_Y\}$, then for all $x \in X$ and $Y \in \mathcal{S}$ we get the equivalence of $(\mathsf{E}\,\mathrm{inj}(v)^{\mathsf{T}})_{x,Y}$ and $x \in Y$. This means, that the elements of $\mathcal{S}$ are represented precisely by the columns of the relation $M := \mathsf{E}\,\mathrm{inj}(v)^{\mathsf{T}} : X \leftrightarrow \mathcal{S}$ since for all $Y \in \mathcal{S}$ it holds $Y = \{x \in X \mid M_x^{(Y)}\}$.

### 3.3 Cartesian Products and Applications

Given a Cartesian product $X \times Y$ of two sets $X$ and $Y$, there are the two canonical projection functions which decompose a pair[1] $u = \langle u_1, u_2 \rangle$ into its first component $u_1$ and its second component $u_2$. For a relation-algebraic approach it is useful to consider instead of these functions the corresponding *projection relations* $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow Y$ such that for all $u \in X \times Y$, $x \in X$ and $y \in Y$ we have $\pi_{u,x}$ if and only if $u_1 = x$ and $\rho_{u,y}$ if and only if $u_2 = y$. Projection relations enable us to specify the well-known pairing operation of functional programming relation-algebraically as follows: For relations $R : Z \leftrightarrow X$ and $S : Z \leftrightarrow Y$ we define their *pairing* (frequently also called *fork* or *tupling*) $[R, S] : Z \leftrightarrow X \times Y$ by

$$[R, S] := R\pi^{\mathsf{T}} \cap S\rho^{\mathsf{T}}. \tag{6}$$

Using (6), for all $z \in Z$ and $u \in X \times Y$ a simple reflection shows that $[R, S]_{z,u}$ if and only if $R_{z,u_1}$ and $S_{z,u_2}$. As a consequence, the *exchange relation*

$$\mathsf{X} := [\rho, \pi] = \rho\pi^{\mathsf{T}} \cap \pi\rho^{\mathsf{T}} \tag{7}$$

of type $[X \times Y \leftrightarrow Y \times X]$ exchanges the components of a pair. This means that for all $u \in X \times Y$ and $v \in Y \times X$ the relationship $\mathsf{X}_{u,v}$ holds if and only if $u_1 = v_2$ and $u_2 = v_1$.

By a combination of the constructions introduced so far, a lot of the well-known operations and predicates on sets can be specified as relations. In the present paper, we need the following:

$$\begin{array}{lll} \mathsf{M} := \mathrm{syq}([\mathsf{E}, \mathsf{E}], \mathsf{E}) & \mathsf{R} := \mathrm{syq}([\overline{\mathsf{I}}, \mathsf{E}], \mathsf{E}) & \mathsf{C} := \mathrm{syq}(\mathsf{E}, \overline{\mathsf{E}}) \\ \mathsf{J} := \mathrm{syq}(\overline{[\overline{\mathsf{E}}, \overline{\mathsf{E}}]}, \mathsf{E}) & \mathsf{A} := \mathrm{syq}(\overline{[\overline{\mathsf{I}}, \overline{\mathsf{E}}]}, \mathsf{E}) & \mathsf{S} := \overline{\mathsf{E}^{\mathsf{T}}\overline{\mathsf{E}}} \end{array} \tag{8}$$

The relations $\mathsf{M}$ and $\mathsf{J}$ have type $[2^X \times 2^X \leftrightarrow 2^X]$ and relation-algebraically specify set intersection and set union, respectively, since for all $\langle S, T \rangle \in 2^X \times 2^X$ and $U \in 2^X$ it holds $\mathsf{M}_{\langle S,T \rangle,U}$ if and only if $S \cap T = U$ and $\mathsf{J}_{\langle S,T \rangle,U}$ if and only if $S \cup T = U$. The type of $\mathsf{R}$ and $\mathsf{A}$ is $[X \times 2^X \leftrightarrow 2^X]$, and these relations specify the removal and addition of elements, respectively. The latter means

[1] In the present paper we always denote the first component of a pair $u \in X \times Y$ by $u_1$ and the second component by $u_2$.

that for all $\langle x,T\rangle \in X\times 2^X$ and $U\in 2^X$ it holds $\mathsf{R}_{\langle x,T\rangle,U}$ if and only if $T\setminus\{x\}=U$ and $\mathsf{A}_{\langle x,T\rangle,U}$ if and only if $T\cup\{x\}=U$. Finally, $\mathsf{C}$ and $\mathsf{S}$ have type $[2^X\leftrightarrow 2^X]$ and for all $S,T\in 2^X$ it holds $\mathsf{C}_{S,T}$ if and only if $T=\overline{S}$ and $\mathsf{S}_{S,T}$ if and only if $S\subseteq T$. Hence, $\mathsf{C}$ specifies set complementation $S\mapsto \overline{S} := X\setminus S$ and $\mathsf{S}$ specifies the subset order. To demonstrate how the relation-algebraic specifications of (8) formally can be developed, we consider the most complicated case of set union. Assume $\langle S,T\rangle\in 2^X\times 2^X$ and $U\in 2^X$. Then we have

$$
\begin{aligned}
S\cup T=U &\iff \forall x\in X:(x\in S\vee x\in T)\leftrightarrow x\in U\\
&\iff \forall x\in X:\neg(x\notin S\wedge x\notin T)\leftrightarrow x\in U\\
&\iff \forall x\in X:\neg(\overline{\mathsf{E}}_{x,S}\wedge \overline{\mathsf{E}}_{x,T})\leftrightarrow \mathsf{E}_{x,U}\\
&\iff \forall x\in X:\overline{[\overline{\mathsf{E}},\overline{\mathsf{E}}]}_{x,\langle S,T\rangle}\leftrightarrow \mathsf{E}_{x,U}\\
&\iff \mathrm{syq}(\overline{[\overline{\mathsf{E}},\overline{\mathsf{E}}]},\mathsf{E})_{\langle S,T\rangle,U},
\end{aligned}
$$

and the definition of the relation $\mathsf{J}$ in (8) shows the desired result.

We end this section with the following two functions (in the usual mathematical sense) which establish a Boolean lattice isomorphism between the two Boolean lattices $[X\leftrightarrow Y]$ and $[X\times Y\leftrightarrow \mathbf{1}]$. In the following equations $\pi : X\times Y\leftrightarrow X$ and $\rho : X\times Y\leftrightarrow Y$ are the projection relations of the underlying Cartesian product and $\mathsf{L}$ is a universal vector of type $[Y\leftrightarrow\mathbf{1}]$.

$$
\mathrm{vec}(R)=(\pi R\cap\rho)\mathsf{L}\qquad\qquad \mathrm{rel}(v)=\pi^{\mathsf{T}}(\rho\cap v\mathsf{L}^{\mathsf{T}}) \tag{9}
$$

The function vec defines the vector $\mathrm{vec}(R)$ corresponding to the relation $R$, and the inverse function rel defines the relation $\mathrm{rel}(v)$ corresponding to the vector $v$. Using a component-wise notation, these definitions say that for all $x\in X$ and $y\in Y$ we have $R_{x,y}$ if and only if $\mathrm{vec}(R)_{\langle x,y\rangle}$ and $v_{\langle x,y\rangle}$ if and only if $\mathrm{rel}(v)_{x,y}$.

## 4 Investigating Simple Games with Relation Algebra

In this section, first we introduce two relation-algebraic models of simple games and show how each of them can be transformed into the other one. Based on the vector model, we then demonstrate how to specify important notions of simple games in the language of relation algebra. All specifications will be algorithmic since they either are relation-algebraic expressions or inclusions respectively equations between such expressions. Hence, they can be executed with the help of RelView after a simple translation into the programming language of this tool.

### 4.1 Relation-Algebraic Models of Simple Games

A first possibility to model a simple game $(N,\mathcal{W})$ with relation-algebraic means is to use a vector $v:2^N\leftrightarrow\mathbf{1}$ that represents the set $\mathcal{W}$ as subset of $2^N$ in the sense of Section 3.2. Frequently, $v$ is called the characteristic vector of the game; in our context we call it the *vector model*. Given such a model $v$, from Section 3.2 we already know that then the columns $M^{(S)}$, $S\in\mathcal{W}$, of the relation $M:=\mathsf{E}\,\mathrm{inj}(v)^{\mathsf{T}} : N\leftrightarrow\mathcal{W}$ precisely represent all winning coalitions. Hence, the game $(N,\mathcal{W})$ can also be modeled by the relation $M$. Since $M$ specifies membership of players in winning coalitions, i.e., $M_{k,S}$ if and only if $k\in S$, for all $k\in N$ and $S\in\mathcal{W}$, we call it the *membership model*.

The definition of $M$ shows how to transform the vector model into the membership model. We formulate this once again as first part of the following theorem, where $\mathsf{E}$ is the is-element relation between players and coalitions. In the second part of the theorem we show how to obtain the vector model back from the membership model.

**Theorem 4.1.1** *Let $(N, \mathcal{W})$ be a simple game. If $v : 2^N \leftrightarrow \mathbf{1}$ is the game's vector model, then $\mathsf{E}\,\mathrm{inj}(v)^{\mathsf{T}} : N \leftrightarrow \mathcal{W}$ is its membership model. Conversely, if $M : N \leftrightarrow \mathcal{W}$ is the game's membership model, then $\mathrm{syq}(\mathsf{E}, M)\mathsf{L} : 2^N \leftrightarrow \mathbf{1}$ (with $\mathsf{L} : \mathcal{W} \leftrightarrow \mathbf{1}$) is its vector model.*

**Proof:** By construction, we have $\mathrm{syq}(\mathsf{E}, M) : 2^N \leftrightarrow \mathcal{W}$. Now, for all $S \in 2^N$ we get

$$\begin{aligned}
(\mathrm{syq}(\mathsf{E}, M)\mathsf{L})_S &\iff \exists\, T \in \mathcal{W} : \mathrm{syq}(\mathsf{E}, M)_{S,T} \wedge \mathsf{L}_T \\
&\iff \exists\, T \in \mathcal{W} : \forall\, k \in N : \mathsf{E}_{k,S} \leftrightarrow M_{k,T} \\
&\iff \exists\, T \in \mathcal{W} : \forall\, k \in N : k \in S \leftrightarrow k \in T \\
&\iff \exists\, T \in \mathcal{W} : S = T \\
&\iff S \in \mathcal{W},
\end{aligned}$$

and this property shows that the vector $\mathrm{syq}(\mathsf{E}, M)\mathsf{L}$ represents $\mathcal{W}$ as subset of $2^N$, as required for the vector model of the game. □

Since the columns of the membership model $M$ of $(N, \mathcal{W})$ enumerate the winning coalitions, with regard to the use of RelView the relation $M$ is appropriate for input and output. Which coalitions are winning hardly can be seen from the vector model $v$. However, as experience has shown, the great advantage of the vector model is that it enables in many cases much more elegant relation-algebraic specifications than the membership model. This holds in particular if a task requires to treat coalitions which are non-winning. For $S \in 2^N$ in the vector model the property $S \notin \mathcal{W}$ is simple expressed by $\overline{v}_S$, whereas in the membership model, for instance, it may require to consider the vector representation $s : N \leftrightarrow \mathbf{1}$ of $S$ and to verify $\mathrm{syq}(M, s) = \mathsf{O}$. Specifications based on the vector model are frequently even more efficient than membership-based ones. This is especially the case if a high percentage of coalitions is winning, since then in the membership model a lot of columns occur. That almost half of the coalitions are winning is typical in practice. E.g., using data from van Deemen (1989, [12]), van Roozendaal (1990, [34]) and Berghammer et al. (2009, [8]), with the help of RelView we obtained for Dutch parliaments that from the 8192 possible coalitions of the 13-parties parliament after the 1972 election 3999 (48.8%) are winning, from the 1024 possible coalitions of the present 10-parties parliament 505 (49.3%) are winning and from the 64 possible coalitions of the 6-parties parliament after the 1986 election even 32 (50%) are winning. However, it should be mentioned here, too, that in cases of specific problems on simple games with larger sets of players but rather small sets of winning coalitions (caused by additional restrictions like connectedness wrt. a policy order or an additional 'accepts a coalition with' relation on the set of players) frequently the membership model allows more efficient solutions than the vector model.

Due to lack of space, apart from the visualization of input and output, in the remainder of the paper we restrict us to the vector model and postpone the presentation of the results concerning the membership model to a future paper. In the next theorem we give first examples for relation-algebraic specifications of game-theoretic notions that base on the vector model. In it, $\mathsf{S} : 2^N \leftrightarrow 2^N$ denotes the subset order as introduced in (8).

**Theorem 4.1.2** *Assume* $v : 2^N \leftrightarrow \mathbf{1}$ *to be the vector model of a simple game* $(N, \mathcal{W})$*. Then* $(N, \mathcal{W})$ *is monotone if and only if* $\mathsf{S}\,\overline{v} \subseteq \overline{v}$ *and is a voting game if and only if in addition* $v \neq \mathsf{O}$ *and* $v \subseteq \mathsf{E}^\mathsf{T}\mathsf{L}$*.*

**Proof:** That $\mathsf{S}\,\overline{v} \subseteq \overline{v}$ specifies monotonicity follows from

$$\begin{aligned}
\mathsf{S}\,\overline{v} \subseteq \overline{v} &\iff \forall\, S \in 2^N : (\mathsf{S}\,\overline{v}\,)_S \rightarrow \overline{v}_S \\
&\iff \forall\, S \in 2^N : (\exists\, T \in 2^N : \mathsf{S}_{S,T} \wedge \overline{v}_T) \rightarrow \overline{v}_S \\
&\iff \forall\, S, T \in 2^N : S \subseteq T \wedge T \notin \mathcal{W} \rightarrow S \notin \mathcal{W} \\
&\iff \forall\, S, T \in 2^N : S \subseteq T \wedge S \in \mathcal{W} \rightarrow T \in \mathcal{W}.
\end{aligned}$$

The equivalence of $v \neq \mathsf{O}$ and $\mathcal{W} \neq \emptyset$ is trivial and the remaining claim is shown by

$$\begin{aligned}
v \subseteq \mathsf{E}^\mathsf{T}\mathsf{L} &\iff \forall\, S \in 2^N : v_S \rightarrow \exists\, k \in N : \mathsf{E}^\mathsf{T}_{S,k} \wedge \mathsf{L}_k \\
&\iff \forall\, S \in 2^N : S \in \mathcal{W} \rightarrow \exists\, k \in N : k \in S \\
&\iff \forall\, S \in 2^N : S \in \mathcal{W} \rightarrow S \neq \emptyset. \qquad \square
\end{aligned}$$

In the next theorem we relation-algebraically specify the properties of a simple game of being proper and strong. Here $\mathsf{C} : 2^N \leftrightarrow 2^N$ is the relation for set complementation; cf. (8).

**Theorem 4.1.3** *Given* $v : 2^N \leftrightarrow \mathbf{1}$ *as the vector model of a simple game* $(N, \mathcal{W})$*, the game is proper if and only if* $v \subseteq \mathsf{C}\,\overline{v}$ *and the game is strong if and only if* $\overline{v} \subseteq \mathsf{C}v$*.*

**Proof:** Starting with a formal logical specification of being a proper game, the first claim is shown by the calculation

$$\begin{aligned}
\forall\, S \in 2^N : S \in \mathcal{W} \rightarrow \overline{S} \notin \mathcal{W} &\iff \forall\, S \in 2^N : S \in \mathcal{W} \rightarrow \exists\, T \in 2^N : T = \overline{S} \wedge T \notin \mathcal{W} \\
&\iff \forall\, S \in 2^N : v_S \rightarrow \exists\, T \in 2^N : \mathsf{C}_{S,T} \wedge \overline{v}_T \\
&\iff v \subseteq \mathsf{C}\,\overline{v}\,.
\end{aligned}$$

In the same way the second specification can be calculated. $\square$

Since $\mathsf{C}$ is a mapping in the relation-algebraic sense, we get due to Schmidt and Ströhlein (1993, [40]) that $\overline{v} \subseteq \mathsf{C}v$ if and only if $\mathsf{C}\,\overline{v} = \overline{\mathsf{C}v} \subseteq v$. Hence, the simple game is decisive (i.e., proper and strong) if and only if $v = \mathsf{C}\,\overline{v}$ if and only if $\overline{v} = \mathsf{C}v$.

Also for weighted majority games a vector model $v : 2^N \leftrightarrow \mathbf{1}$ can be computed within relation algebra. To this end, the players are interpreted as the parties of a parliament and the weights are interpreted as the number of the parliament seats the party holds, i.e., in the very same way as in real political life. Then the only requirement to obtain $v$ is that, given $X$ as set of seats, there is a mapping (in the relation-algebraic sense) $D : X \leftrightarrow N$ at hand that describes the distribution of the seats, i.e., fulfills for all $x \in X$ and $k \in N$ that $D_{x,k}$ if and only if seat $x$ is owned by party $k$. Since the concrete procedure is irrelevant for the remainder of the paper, we don't go into details here and refer the interested reader to Berghammer et al. (2009, [8]), where the computation of $v$ from $D$ formally is developed.

In general, the number of winning coalitions of a simple game can grow rapidly with the number of players. For example, if the game is proper and strong and $n$ is the number of players, then the number of winning coalitions is $2^{n-1}$, i.e., 50% of all coalitions are winning. Therefore, in the following example that shall demonstrate the visualization facilities of RelView we deal with a rather small game, taken from Lorenzo-Freire et al. (2007, [26]).

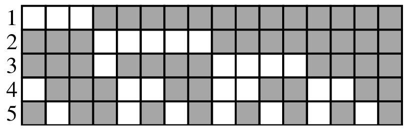

**Fig. 1.** Membership model of the Catalonian game

**Example 4.1.1.** We consider the following weighted majority game with five players, that is a representation of the parliament of Catalonia, one of the 17 Spanish autonomous communities, after the November 2003 election.

$$[68; 46, 42, 23, 15, 9]$$

The players are, from left to right, labeled with the numbers 1, 2, 3, 4 and 5; they correspond (in the same order) to the five Catalonian parties CIU, PSC-CPC, ERC, PP and ICV-EA. In the picture of Figure 1 the membership model $M : N \leftrightarrow \mathcal{W}$ of this game is shown as depicted by RelView in the relation-window of its user interface. In this $5 \times 16$ Boolean matrix a black square means a 1-entry and a white square means a 0-entry. So, for example, the winning coalition represented by the first column of $M$ consists of the three parties PSC-CPC, ERC and ICV-EA. If we transform the membership model $M$ into the vector model, we obtain a vector $v : 2^N \leftrightarrow \mathbf{1}$ in which exactly 16 entries are 1. The two pictures of Figure 2 show the is-element relation $\mathsf{E} : N \leftrightarrow 2^N$ and, below it, the transpose of the vector $v$ (that is, the row vector $v^\mathsf{T} : \mathbf{1} \leftrightarrow 2^N$). The 32 columns of the is-element relation $\mathsf{E}$ represent the 32 coalitions. A comparison of the pictures (here the row vector representation of the game is of great advantage) shows that the 1-entries of the vector model $v$ precisely designate those columns of $\mathsf{E}$ that belong to the membership model $M$.

As a weighted majority game, $(N, \mathcal{W})$ is monotone. We also have tested whether it is proper and strong using the RelView-versions

```
proper(E,v) = incl(v,Compl(E)*-v)     stronc(E,v) = incl(-v,Compl(E)*v)
```

of the specifications of Theorem 4.1.3, where the pre-defined RelView-operation `incl` tests inclusion of relations and the RelView-function

```
Compl(E) = syq(-E,-E)
```

computes from the is-element relation $\mathsf{E} : N \leftrightarrow 2^N$ the relation $\mathsf{S} : 2^N \leftrightarrow 2^N$ for set complementation. In both cases we obtained the answer 'yes'. $\Box$

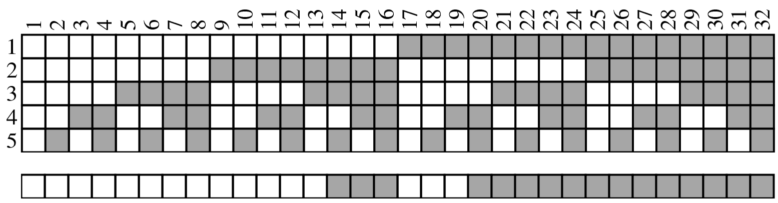

**Fig. 2.** Vector model of the Catalonian game

## 4.2 Computing Minimal Coalitions and Related Notions

Computing minimal winning coalitions with relation-algebraic means is easy. It is well-known, cf. Schmidt and Ströhlein (1993, [40]) that, given a strict order relation $R : X \leftrightarrow X$ and a vector $w : X \leftrightarrow \mathbf{1}$ that represents a subset $Y$ of $X$, the vector $w \cap \overline{R^{\mathsf{T}} w} : X \leftrightarrow \mathbf{1}$ represents the set of minimal elements of $Y$ as a subset of $X$. Hence, if we take $w$ as vector model $v : 2^N \leftrightarrow \mathbf{1}$ of a simple game $(N, \mathcal{W})$ and $R$ as the irreflexive part of the subset order $\mathsf{S} : 2^N \leftrightarrow 2^N$, we immediately get the following result.

**Theorem 4.2.1** *If $v : 2^N \leftrightarrow \mathbf{1}$ is the vector model of the simple game $(N, \mathcal{W})$, then the vector*

$$\mathrm{minwin}(v) := v \cap \overline{(\mathsf{S} \cap \overline{\mathsf{I}})^{\mathsf{T}} v}$$

*of type $[2^N \leftrightarrow \mathbf{1}]$ represents the set $\mathcal{W}_{\min}$ of minimal winning coalitions.* □

In the next theorem we relation-algebraically specify the is-swinger relation and, based on it, the vector of vulnerable winning coalitions. To simplify the calculations, we only consider monotone games. With regard to practical applications this is no serious restriction.[2] Recall from Section 3, that $\mathsf{R}$ is the relation-algebraic specification of element-removal and the function rel yields for a vector that represents a subset of a Cartesian product in the sense of Section 3.2 the corresponding 'proper' relation.

**Theorem 4.2.2** *Let $v : 2^N \leftrightarrow \mathbf{1}$ be the vector model of a monotone simple game $(N, \mathcal{W})$. If we define the relation* $\mathrm{Swingers}(v) : N \leftrightarrow 2^N$ *and the vector* $\mathrm{vulwin}(v) : 2^N \leftrightarrow \mathbf{1}$ *by*

$$\mathrm{Swingers}(v) := \mathsf{E} \cap \mathsf{L} v^{\mathsf{T}} \cap \mathrm{rel}(\mathsf{R}\,\overline{v}\,) \qquad \mathrm{vulwin}(v) := \mathrm{Swingers}(v)^{\mathsf{T}} \mathsf{L}$$

*(with $\mathsf{L} : N \leftrightarrow \mathbf{1}$), then for all $k \in N$ and $S \in 2^N$ it holds* $\mathrm{Swingers}(v)_{k,S}$ *if and only if $k$ is a swinger of $S$ and* $\mathrm{vulwin}(v)_S$ *if and only if $S$ is a vulnerable winning coalition.*

**Proof:** For all $k \in N$ and $S \in 2^N$ we have

$$\begin{aligned}
\mathrm{Swingers}(v)_{k,S} &\iff (\mathsf{E} \cap \mathsf{L} v^{\mathsf{T}} \cap \mathrm{rel}(\mathsf{R}\,\overline{v}\,))_{k,S} \\
&\iff \mathsf{E}_{k,S} \wedge (\mathsf{L} v^{\mathsf{T}})_{k,S} \wedge \mathrm{rel}(\mathsf{R}\,\overline{v}\,)_{k,S} \\
&\iff \mathsf{E}_{k,S} \wedge (\mathsf{L} v^{\mathsf{T}})_{k,S} \wedge (\mathsf{R}\,\overline{v}\,)_{\langle k,S\rangle} \\
&\iff \mathsf{E}_{k,S} \wedge (\mathsf{L} v^{\mathsf{T}})_{k,S} \wedge \exists\, T \in 2^N : \mathsf{R}_{\langle k,S\rangle,T} \wedge \overline{v}_{\,T} \\
&\iff \mathsf{E}_{k,S} \wedge v_S \wedge \exists\, T \in 2^N : S \setminus \{k\} = T \wedge \overline{v}_{\,T} \\
&\iff \mathsf{E}_{k,S} \wedge v_S \wedge \overline{v}_{\,S\setminus\{k\}} \\
&\iff k \in S \wedge S \in \mathcal{W} \wedge S \setminus \{k\} \notin \mathcal{W}.
\end{aligned}$$

Since $(N, \mathcal{W})$ is monotone, the last formula specifies $k$ as a swinger of $S$. This is the first result. Using it, the second one is shown by

$$\begin{aligned}
\mathrm{vulwin}(v)_S &\iff (\mathrm{Swingers}(v)^{\mathsf{T}} \mathsf{L})_S \\
&\iff \exists\, k \in N : \mathrm{Swingers}(v)^{\mathsf{T}}_{S,k} \wedge \mathsf{L}_k \\
&\iff \exists\, k \in N : k \in S \wedge S \in \mathcal{W} \wedge S \setminus \{k\} \notin \mathcal{W} \\
&\iff S \in \mathcal{W} \wedge \exists\, k \in N : k \in S \wedge S \setminus \{k\} \notin \mathcal{W}.
\end{aligned}$$ □

[2] In the political science literature typically one only considers monotone simple games as, e.g., in Peleg (1981, [31]), or even demands a simple game to be monotone by definition as, e.g., van Deemen (1989, [12]) and van Roozendaal (1990, [34]) do.

So far, we have considered specific kinds of coalitions. In the remainder of the section, we turn towards specific players with more or less power such as a *dummy*, a *vetoer*, a *dictator*, and a *null player*. The next theorem shows how the sets of these specific players relation-algebraically can be specified as vectors. It uses the relation $\mathsf{A}$ of (8) for the addition of an element.

**Theorem 4.2.3** *Based on the vector model* $v : 2^N \leftrightarrow \mathbf{1}$ *of a simple game and* $m := \mathrm{minwin}(v)$ *as vector representation of the set* $\mathcal{W}_{\min}$ *of minimal winning coalitions, we consider the following four vectors of type* $[N \leftrightarrow \mathbf{1}]$ *(where* $[N \leftrightarrow N]$ *is the type of the* $\mathsf{I}$ *in* $\mathrm{syq}(\mathsf{I}, \mathsf{E})$ *and* $[2^N \leftrightarrow 2^N]$ *is the type of the* $\mathsf{I}$ *in* $\overline{\overline{\mathsf{I}}\,m}$*):*

$$\mathrm{dummy}(m) := \overline{\mathsf{E}m} \qquad \mathrm{vetoer}(m) := \overline{\overline{\mathsf{E}}\,m}$$

$$\mathrm{dictator}(m) := \mathrm{syq}(\mathsf{I}, \mathsf{E})(m \cap \overline{\overline{\mathsf{I}}\,m}\,) \qquad \mathrm{null}(v) := \mathrm{syq}(\mathrm{rel}(\mathsf{A}v)^{\mathsf{T}}, v)$$

*Then* $\mathrm{dummy}(m)$ *(*$\mathrm{vetoer}(m)$*,* $\mathrm{dictator}(m)$ *and* $\mathrm{null}(v)$*, respectively) represents the set of dummies (vetoers, dictators and null players, respectively).*

**Proof:** We only verify the specifications for dictators and null players. Assume $k \in N$. Then the first case follows from

$$\begin{aligned}
\mathrm{dictator}(m)_k &\iff (\mathrm{syq}(\mathsf{I}, \mathsf{E})(m \cap \overline{\overline{\mathsf{I}}\,m}\,))_k \\
&\iff \exists\, S \in 2^N : \mathrm{syq}(\mathsf{I}, \mathsf{E})_{k,S} \wedge m_S \wedge \overline{\overline{\mathsf{I}}\,m}_{\,S} \\
&\iff \exists\, S \in 2^N : (\forall\, j \in N : \mathsf{I}_{j,k} \leftrightarrow \mathsf{E}_{j,S}) \wedge m_S \wedge \neg\exists\, T \in \mathcal{W} : \overline{\mathsf{I}}_{\,S,T} \wedge m_T \\
&\iff \exists\, S \in 2^N : (\forall\, j \in N : j = k \leftrightarrow j \in S) \wedge m_S \wedge \forall\, T \in \mathcal{W} : m_T \rightarrow S = T \\
&\iff \exists\, S \in 2^N : S = \{k\} \wedge S \in \mathcal{W}_{\min} \wedge \forall\, T \in \mathcal{W}_{\min} : S = T
\end{aligned}$$

and the second case follows from

$$\begin{aligned}
\mathrm{null}(v)_k &\iff \mathrm{syq}(\mathrm{rel}(\mathsf{A}v)^{\mathsf{T}}, v)_k \\
&\iff \forall\, S \in 2^N : \mathrm{rel}(\mathsf{A}v)^{\mathsf{T}}_{S,k} \leftrightarrow v_S \\
&\iff \forall\, S \in 2^N : (\mathsf{A}v)_{\langle k,S\rangle} \leftrightarrow v_S \\
&\iff \forall\, S \in 2^N : (\exists\, T \in 2^N : \mathsf{A}_{\langle k,S\rangle,T} \wedge v_T) \leftrightarrow v_S \\
&\iff \forall\, S \in 2^N : (\exists\, T \in 2^N : S \cup \{k\} = T \wedge T \in \mathcal{W}) \leftrightarrow S \in \mathcal{W} \\
&\iff \forall\, S \in 2^N : S \cup \{k\} \in \mathcal{W} \leftrightarrow S \in \mathcal{W},
\end{aligned}$$

since in both cases the last formula of the derivation is the formal logical specification of the property under consideration. □

In a simple game at most one dictator can exist. Hence, if $\mathrm{dictator}(m)$ is not empty, then this vector is a point in the sense of Section 3.2. Using the above specifications, it immediately can be tested whether a simple game is *weak* or *dictatorial*, respectively, since, translated into relation algebra, the first property means that $\mathrm{vetoer}(m) \neq \mathsf{O}$ and the second property means that $\mathrm{dictator}(m) \neq \mathsf{O}$.

Finally, let us consider what the RelView-programs corresponding to the specifications of this subsection yield in the case of our running example.

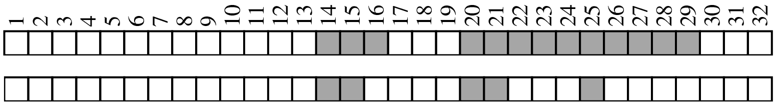


**Fig. 3.** Vulnerable and minimal winning coalitions of the Catalonian game

**Example 4.2.1.** If the RelView-programs we have obtained from the relation-algebraic specifications vulwin$(v)$ and minwin$(v)$ of Theorem 4.2.2 and Theorem 4.2.1[3] are applied to the vector model of Example 4.1.1, then the tool yields two vectors which, again in transposed form to save space, are shown in the two pictures of Figure 3. The row vector on the top designates the 13 vulnerable winning coalitions of the parliament of Catalonia after the 2003 election, and that under it designates the five minimal winning coalitions. From these results we could obtain the 'concrete' form of the coalitions by a comparison with the columns of the is-element relation $\mathsf{E} : N \leftrightarrow 2^N$ as remarked in Example 4.1.1. The much more easier way is, however, to use the technique for the column-wise enumeration of sets presented in Section 3.2, i.e., to evaluate the two expressions $\mathsf{E}\,\mathrm{vulwin}(v)^{\mathsf{T}}$ and $\mathsf{E}\,\mathrm{minwin}(v)^{\mathsf{T}}$. Doing so, we obtain the left-most and right-most of the three RelView-matrices of Figure 4, from which each vulnerable winning coalition and each minimal winning coalition, respectively, immediately can be obtained as a column.

The RelView-matrix in the middle column-wisely enumerates the sets of swingers of the vulnerable winning coalitions. It is obtained by removing from the is-swinger relation all columns corresponding to a 0-entry in the vector representation of the vulnerable winning coalitions. Relation-algebraically this reads as $\mathrm{Swingers}(v)\,\mathrm{inj}(\mathrm{vulwin}(v))^{\mathsf{T}}$.

To explain the three RelView-matrices of Figure 4 a bit more, we compare the first columns of the two $5 \times 13$ matrices. Since they are identical, that means, each party is a swinger, the represented coalition $\{$PSC-CPC, ERC, ICV-EA$\}$ is a minimal winning one. This agrees with the column-wise enumeration of these coalitions in which the coalition appears, too. Next, we compare the third columns of the two matrices. From the first matrix we get $\{$PSC-CPC, ERC, PP, ICV-EA$\}$ as vulnerable winning coalition and from the second one $\{$PSC-CPC, ERC$\}$ as the set of its swingers. Hence, this coalition is not minimal winning. Again this agrees with the right-most matrix, since now the coalition does not occur as a column.

To demonstrate RelView's visualization potential a bit more, the RelView-graph of Figure 5 shows the Hasse-diagram of the inclusion order $\mathsf{S}$ of the 32 coalitions of our example. In this picture the inclusion relationships between the 16 winning coalitions are highlightened by boldface arcs; from this it immediately becomes clear that the game is monotone. The five minimal winning coalitions are drawn as white squares and the 11 non-minimal winning coalitions are drawn as black circles.

[3] Since the is-swinger relation is very decisive for computing power indices, we postpone its presentation until Section 4.4. That section is devoted to the computation of power indices.

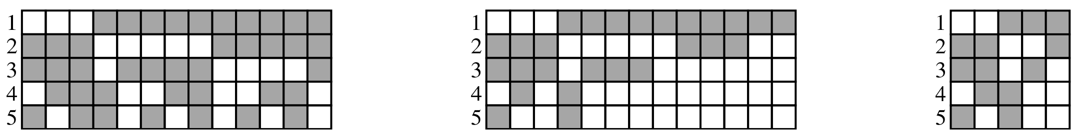


**Fig. 4.** Column-wise enumeration of the vulnerable and minimal winning coalitions

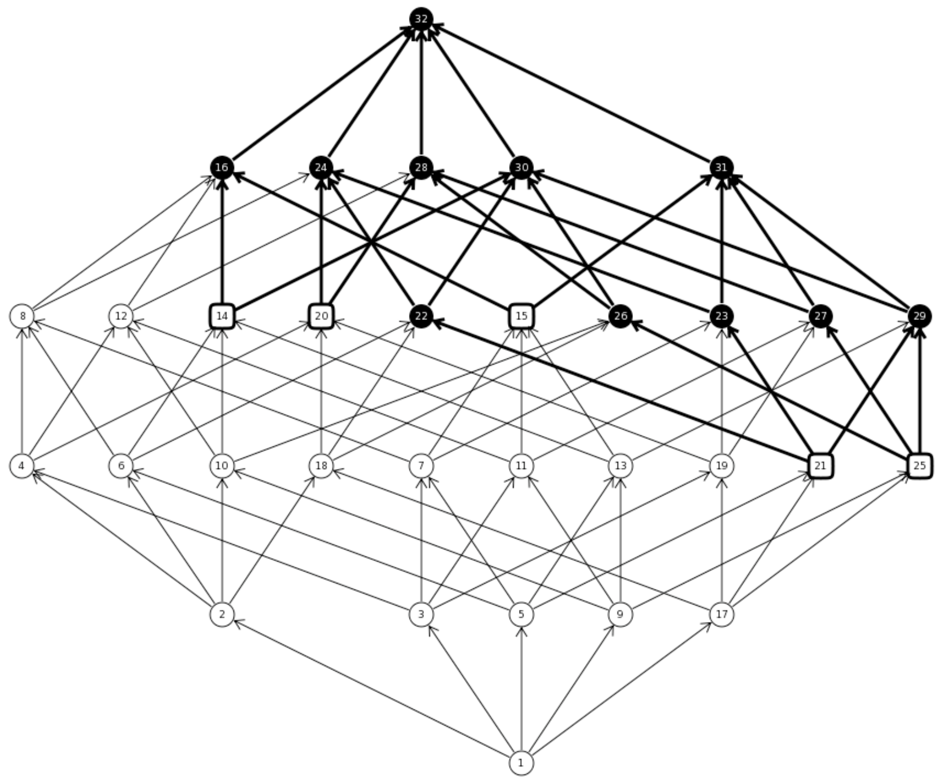


**Fig. 5.** The ordered coalitions of the Catalonian game

For our running example we also have computed the vectors specified in Theorem 4.2.3. Here all results the RelView tool delivered were empty. □

### 4.3 Computing Central and Dominant Players

In this section we deliver relation-algebraic specifications of the sets of central and dominant players. Let us start with the concept of a central player. Note that since there exists at most one central player in a simple game, the vector given in the following theorem either is empty or is a point in the sense of Section 3.2.

**Theorem 4.3.1** *Let a simple game $(N, \mathcal{W})$ with a policy order $P : N \leftrightarrow N$ be given and assume that $v : 2^N \leftrightarrow \mathbf{1}$ is the game's vector model. Then the vector*

$$\mathrm{central}(v, P) := \overline{\mathrm{syq}(P, \mathsf{E})v} \cap \overline{\mathrm{syq}(P^\mathsf{T}, \mathsf{E})v} \cap \mathrm{syq}(P \cup \mathsf{I}, \mathsf{E})v \cap \mathrm{syq}(P^\mathsf{T} \cup \mathsf{I}, \mathsf{E})v$$

*of type $[N \leftrightarrow \mathbf{1}]$ (where $\mathsf{I} : N \leftrightarrow N$) represents the set of central players.*

**Proof:** Let $k \in N$ be a player. Then we have

$$\begin{aligned}
\overline{\mathrm{syq}(P, \mathsf{E})v}_k &\iff \neg \exists\, S \in 2^N : \mathrm{syq}(P, \mathsf{E})_{k,S} \wedge v_S \\
&\iff \forall\, S \in 2^N : (\forall\, j \in N : P_{j,k} \leftrightarrow \mathsf{E}_{j,S}) \to \overline{v}_S \\
&\iff \forall\, S \in 2^N : (\forall\, j \in N : P_{j,k} \leftrightarrow j \in S) \to \overline{v}_S \\
&\iff \forall\, S \in 2^N : S = \{j \in N : P_{j,k}\} \to S \notin \mathcal{W} \\
&\iff \{j \in N : P_{j,k}\} \notin \mathcal{W}
\end{aligned}$$

and a replacement of $P$ by its transpose in this calculation shows

$$\overline{\mathrm{syq}(P^{\mathsf{T}},\mathsf{E})v} \iff \{j \in N : P_{k,j}\} \notin \mathcal{W}.$$

Next, we deal with the third expression of the intersection and get

$$\begin{aligned}
(\mathrm{syq}(P \cup \mathsf{I},\mathsf{E})v)_k &\iff \exists\, S \in 2^N : \mathrm{syq}(P \cup \mathsf{I},\mathsf{E})_{k,S} \wedge v_S \\
&\iff \exists\, S \in 2^N : (\forall\, j \in N : (P_{j,k} \vee \mathsf{I}_{j,k}) \leftrightarrow \mathsf{E}_{j,S}) \wedge v_S \\
&\iff \exists\, S \in 2^N : (\forall\, j \in N : (P_{j,k} \vee j = k) \leftrightarrow j \in S) \wedge v_S \\
&\iff \exists\, S \in 2^N : S = \{j \in N : P_{j,k}\} \cup \{k\} \wedge S \in \mathcal{W} \\
&\iff \{j \in N : P_{j,k}\} \cup \{k\} \in \mathcal{W}.
\end{aligned}$$

Again by a replacement of $P$ by $P^{\mathsf{T}}$ we find for the fourth expression

$$(\mathrm{syq}(P^{\mathsf{T}} \cup \mathsf{I},\mathsf{E})v)_k \iff \{j \in N : P_{k,j}\} \cup \{k\} \in \mathcal{W}.$$

Finally, the conjunction of the right-hand sides of the derived equivalences precisely means that $k$ is a central player. □

Next, let us study the concept of a dominant player. In the following, we show how two desirability concepts introduced in Peleg (1981, [31]) relation-algebraically can be specified. We do it again in such a way that this leads to RelView-programs after a simple translation step.

In the decisive first part of the following theorem it is shown how the concept 'as-least-as-desirable' relation-algebraically can be specified by means of a vector with a Cartesian product as domain. The – again vector-based – specifications of 'more-desirable' and 'dominance' then are easy consequences of the theorem's first part. Recall from Section 3, that $\mathsf{J}$ and $\mathsf{R}$ are the relation-algebraic specification of set union and element-removal, respectively, $\mathsf{X}$ is the relation for exchanging the components of pairs, the function rel transforms vector representations into 'proper' relations and the function vec is the inverse of rel.

**Theorem 4.3.2** *Let $v : 2^N \leftrightarrow \mathbf{1}$ be the vector model of a simple game $(N,\mathcal{W})$. Then the vector*

$$\mathrm{alades}(v) := \overline{\mathsf{L}([\,\overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}\,,\, \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}\,] \cap [\mathrm{rel}(\mathsf{J}\,\overline{v}\,), \mathrm{rel}(\mathsf{J}v)])}$$

*of type $[2^N \times 2^N \leftrightarrow \mathbf{1}]$ (where $\mathsf{L} : \mathbf{1} \leftrightarrow 2^N$) represents the at-least-as-desirable relation $\geq_D$ as subset of $2^N \times 2^N$. For the more-desirable relation $>_D$ the same is obtained by the vector*

$$\mathrm{mdes}(v) := \mathrm{alades}(v) \cap \overline{\mathsf{X}\,\mathrm{alades}(v)}$$

*of type $[2^N \times 2^N \leftrightarrow \mathbf{1}]$. With $\pi : N \times 2^N \leftrightarrow N$ as first projection of $N \times 2^N$, finally, the vector*

$$\mathrm{dom}(v) := \mathrm{vec}(\mathsf{E}) \cap [\pi\,\mathrm{syq}(\mathsf{I},\mathsf{E}), \mathsf{R}]\,\mathrm{mdes}(v)$$

*of type $[N \times 2^N \leftrightarrow \mathbf{1}]$ represents the dominance relation $\gg$ as subset of $N \times 2^N$.*

**Proof:** Let a pair $\langle S,T\rangle \in 2^N \times 2^N$ be given. We start with the following equivalence:

$$\begin{aligned}
\mathrm{alades}(v)_{\langle S,T\rangle} &\iff \overline{\mathsf{L}([\,\overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}\,,\, \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}\,] \cap [\mathrm{rel}(\mathsf{J}\,\overline{v}\,), \mathrm{rel}(\mathsf{J}v)])}_{\,\langle S,T\rangle} \\
&\iff \neg\exists\, U \in 2^N : \mathsf{L}_{\perp,U} \wedge [\,\overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}\,,\, \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}\,]_{U,\langle S,T\rangle} \wedge [\mathrm{rel}(\mathsf{J}\,\overline{v}\,), \mathrm{rel}(\mathsf{J}v)]_{U,\langle S,T\rangle} \\
&\iff \neg\exists\, U \in 2^N : \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,S} \wedge \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,T} \wedge \mathrm{rel}(\mathsf{J}\,\overline{v}\,)_{U,S} \wedge \mathrm{rel}(\mathsf{J}v)_{U,T} \\
&\iff \neg\exists\, U \in 2^N : \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,S} \wedge \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,T} \wedge (\mathsf{J}\,\overline{v}\,)_{\langle U,S\rangle} \wedge (\mathsf{J}v)_{\langle U,T\rangle} \\
&\iff \forall\, U \in 2^N : \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,S} \wedge \overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,T} \wedge (\mathsf{J}v)_{\langle U,T\rangle} \rightarrow \neg(\mathsf{J}\,\overline{v}\,)_{\langle U,S\rangle}
\end{aligned}$$

Now, we consider the four relationships of the body of the universal quantification. In the first case we calculate

$$\overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,S} \iff \neg\exists\, j \in N : \mathsf{E}^{\mathsf{T}}_{U,j} \wedge \mathsf{E}_{j,S} \iff \neg\exists\, j \in N : j \in U \wedge j \in S \iff U \cap S = \emptyset.$$

In the same way we get the equivalence of the relationship $\overline{\mathsf{E}^{\mathsf{T}}\mathsf{E}}_{U,T}$ and $U \cap T = \emptyset$. For the third relationship we obtain

$$\begin{aligned} (\mathsf{J}v)_{\langle U,T\rangle} &\iff \exists\, V \in 2^N : \mathsf{J}_{\langle U,T\rangle,V} \wedge v_V \\ &\iff \exists\, V \in 2^N : U \cup T = V \wedge V \in \mathcal{W} \\ &\iff U \cup T \in \mathcal{W}. \end{aligned}$$

A similar calculation shows that $(\mathsf{J}\,\overline{v}\,)_{\langle U,S\rangle}$ if and only if $U \cup S \notin \mathcal{W}$. Summing up, we have shown the equivalence

$$\mathrm{alades}(v)_{\langle S,T\rangle} \iff \forall\, U \in 2^N : U \cap S = \emptyset \wedge U \cap T = \emptyset \wedge U \cup T \in \mathcal{W} \rightarrow U \cup S \in \mathcal{W},$$

the right-hand side of which is the formal logical specification of the relationship $S \geq_D T$ and, thus, concludes the proof of the first claim.

To verify the second claim we assume again a pair $\langle S,T\rangle \in 2^N{\times}2^N$ to be given. Then the desired result is shown by the following derivation, since the last line of it is the formal logical specification of $S >_D T$:

$$\begin{aligned} \mathrm{mdes}(v)_{\langle S,T\rangle} &\iff (\mathrm{alades}(v) \cap \overline{\mathsf{X}\,\mathrm{alades}(v)}\,)_{\langle S,T\rangle} \\ &\iff \mathrm{alades}(v)_{\langle S,T\rangle} \wedge \overline{\mathsf{X}\,\mathrm{alades}(v)}_{\langle S,T\rangle} \\ &\iff S \geq_D T \wedge \neg\exists\, \langle U,V\rangle \in 2^N{\times}2^N : \mathsf{X}_{\langle S,T\rangle,\langle U,V\rangle} \wedge \mathrm{alades}(v)_{\langle U,V\rangle} \\ &\iff S \geq_D T \wedge \neg\exists\, \langle U,V\rangle \in 2^N{\times}2^N : S = V \wedge T = U \wedge \mathrm{alades}(v)_{\langle U,V\rangle} \\ &\iff S \geq_D T \wedge \neg\mathrm{alades}(v)_{\langle T,S\rangle} \\ &\iff S \geq_D T \wedge \neg(T \geq_D S) \end{aligned}$$

Finally, the last claim is shown by the following calculation for all pairs $\langle k,S\rangle \in N{\times}2^N$, which uses the equivalence of $(\mathrm{syq}(\mathsf{I},\mathsf{E})_{k,T}$ and $\{k\} = T$ and ends with the logical formula that specifies the relationship $k \gg S$:

$$\begin{aligned} \mathrm{dom}(v)_{\langle k,S\rangle} &\iff (\mathrm{vec}(\mathsf{E}) \cap [\pi\,\mathrm{syq}(\mathsf{I},\mathsf{E}),\mathsf{R}]\,\mathrm{mdes}(v))_{\langle k,S\rangle} \\ &\iff \mathrm{vec}(\mathsf{E})_{\langle k,S\rangle} \wedge ([\pi\,\mathrm{syq}(\mathsf{I},\mathsf{E}),\mathsf{R}]\,\mathrm{mdes}(v))_{\langle k,S\rangle} \\ &\iff \mathsf{E}_{k,S} \wedge \exists\, \langle T,U\rangle \in 2^N{\times}2^N : [\pi\,\mathrm{syq}(\mathsf{I},\mathsf{E}),\mathsf{R}]_{\langle k,S\rangle,\langle T,U\rangle} \wedge \mathrm{mdes}(v)_{\langle T,U\rangle} \\ &\iff k \in S \wedge \exists\, \langle T,U\rangle \in 2^N{\times}2^N : (\mathrm{syq}(\mathsf{I},\mathsf{E})_{k,T} \wedge \mathsf{R}_{\langle k,S\rangle,U} \wedge T >_D U \\ &\iff k \in S \wedge \exists\, \langle T,U\rangle \in 2^N{\times}2^N : \{k\} = T \wedge S \setminus \{k\} = U \wedge T >_D U \\ &\iff k \in S \wedge \{k\} >_D S \setminus \{k\} \end{aligned}$$

□

If we apply the function rel of (9) to the three vectors of Theorem 4.3.2, then we obtain again relation-algebraic specifications rel(alades($v$)), rel(mdes($v$)) and rel(dom($v$)) for the relations $\geq_D$, $>_D$ and $\gg$, respectively, but now as 'proper' relations of type $[2^N \leftrightarrow 2^N]$ in the first two cases and $[N \leftrightarrow 2^N]$ in the latter case. The RelView-versions of rel(alades($v$)) and rel(mdes($v$)) allow to visualize the at-least-as-desirable and the more-desirable relation of a simple game $(N,\mathcal{W})$ with vector model $v : 2^N \leftrightarrow \mathbf{1}$ not only as Boolean matrices but also as directed graphs.

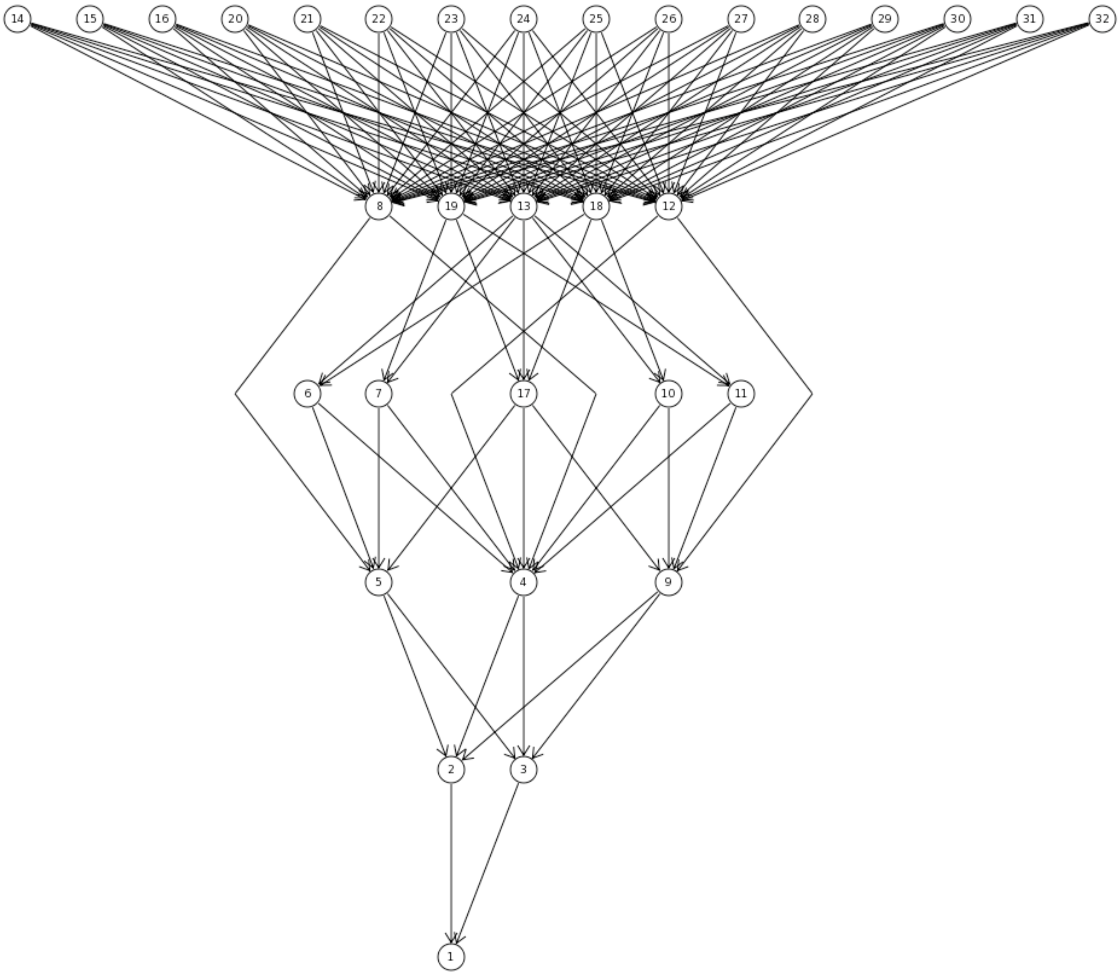


**Fig. 6.** The more-desirable relation of the Catalonian game

As we have already demonstrated, in the latter case additionally features are provided which allow to draw graphs nicely and to highlight selected portions. The specification rel(dom($v$)) at once leads to a RelView-program for determining the game's dominant players. Recall that the dominant players are those which are related to an element of $\mathcal{W}$ via the dominance relation $\gg$. Taking $v$ as vector representation of $\mathcal{W}$ and rel(dom($v$)) as relation-algebraic specification of $\gg$, this immediately yields $\mathrm{rel}(\mathrm{dom}(v))v : N \leftrightarrow \mathbf{1}$ as vector representation of the set of dominant players of the game.

**Example 4.3.1.** Unfortunately, nowhere in the literature a policy order of the parties of our running example seems to have been published. Therefore, in the following we only demonstrate how RelView can be used to treat the concepts of 'desirability' and 'dominance'.

In the RelView-picture of Figure 6 we show the Hasse-diagram of the more-desirable relation rel(mdes($v$)) of the parliament of Catalonia after the 2003 election. The directed graph is drawn using the level-oriented graph-drawing algorithm of Gansner et al. (1993, [18]). From the level at the top we get that half of the coalitions is maximal with respect to 'more-desirability' and these coalitions coincide with the winning ones (since the row vector representation of the winning coalitions of Example 4.1.1 says that precisely the columns of $\mathsf{E}$ with

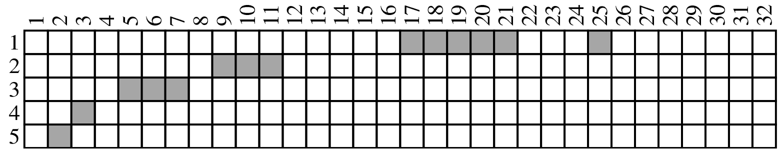


**Fig. 7.** The dominance relation of the Catalonian game

labels 14-16 or labels 20-32 represent winning coalitions). In case of the dominance-relation $\mathrm{rel}(\mathrm{dom}(v)) : N \leftrightarrow 2^N$, the RelView tool delivers the $5 \times 32$ Boolean matrix of Figure 7. It shows that party CIU (row number 1 in the matrix) is the only dominant player of the parliament because it dominates the three winning coalitions with column labels 20, 21 and 25. Recall from the RelView-picture of the vector model in Example 4.1.1 that the coalitions with column labels 17-19 and the coalitions dominated by the other parties are not winning. □

### 4.4 Computing Power Indices

In this section we apply relation algebra to some power indices. More precisely, we present relation-algebraic specifications of the Banzhaf, Holler-Packel, and Deegan-Packel indices.

There is a very close relationship between the relation $\mathrm{Swingers}(v) : N \leftrightarrow 2^N$ of Theorem 4.2.2 and the power indices introduced in (2) and (3) that also is the key for their computation using the RelView tool. This relationship is presented in the next theorem. To enhance readability, in it we denote for $X$ and $Y$ being finite, for $R : X \leftrightarrow Y$ and $x \in X$, the number of 1-entries of $R$ with $|R|$ and the number of 1-entries of the $x$-row of $R$ with $|R|_x$. Hence, $|R|$ equals the cardinality of $R$ (as set of pairs) and $|R|_x$ equals the cardinality of the subset $Y'$ of $Y$ that is represented by the transpose of the $x$-row, i.e. by the vector $(R^{(x)})^\mathsf{T} : Y \leftrightarrow \mathbf{1}$, in the sense of Section 3.2.

**Theorem 4.4.1** *Assume a monotone simple game $(N, \mathcal{W})$ with $n$ players and its vector model $v : 2^N \leftrightarrow \mathbf{1}$. Furthermore, let a player $k \in N$ be given. Then we have for the Banzhaf index that*

$$\text{(i)}\quad \overline{B}(k) = \frac{|\mathrm{Swingers}(v)|_k}{2^{n-1}} \qquad\qquad \text{(ii)}\quad B(k) = \frac{|\mathrm{Swingers}(v)|_k}{|\mathrm{Swingers}(v)|}$$

*and for the Holler-Packel index that*

$$\text{(iii)}\quad \overline{H}(k) = \frac{|\mathrm{Swingers}(\mathrm{minwin}(v))|_k}{|\mathrm{minwin}(v)|} \qquad\qquad \text{(iv)}\quad H(k) = \frac{|\mathrm{Swingers}(\mathrm{minwin}(v))|_k}{|\mathrm{Swingers}(\mathrm{minwin}(v))|}.$$

**Proof:** Equation (i) is trivial since the transpose of the $k$-row of $\mathrm{Swingers}(v)$ represents the set $\{S \in \mathcal{W} \mid k \text{ swinger of } S\}$. Combining it with the definition of $B(k)$, we get

$$B(k) = \frac{\overline{B}(k)}{\sum_{j \in N} \overline{B}(j)} = \frac{\frac{1}{2^{n-1}} |\mathrm{Swingers}(v)|_k}{\frac{1}{2^{n-1}} \sum_{j \in N} |\mathrm{Swingers}(v)|_j} = \frac{|\mathrm{Swingers}(v)|_k}{|\mathrm{Swingers}(v)|}$$

which is (ii). Equation (iii) is again trivial and (iv) is shown analogously to (ii). □

If the RelView tool depicts a relation $R$ as Boolean matrix in the relation-window, then in the window's status bar the number of 1-entries of $R$ is shown. Furthermore, it is able to mark its

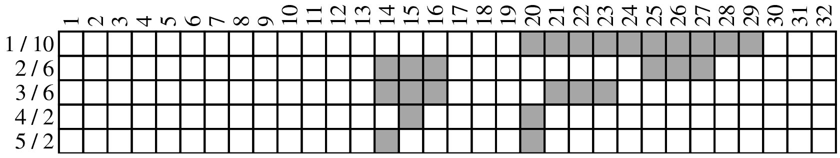


**Fig. 8.** The is-swinger relation of the Catalonian game

rows and columns for explanatory purposes. So far, we only have shown the possibility to attach consecutive row and/or column numbers. But also the numbers of 1-entries can be attached as labels. In combination with Theorem 4.4.1 this immediately allows to compute Banzhaf and Holler-Packel indices. We demonstrate this by means of our running example.

**Example 4.4.1.** If we use RelView to compute the is-swinger relation Swingers($v$) for the vector model $v$ of our running Catalonia parliament example and additionally instruct the tool to attach consecutive row and column numbers, and for each row also the number of its 1-entries as second label (after the sign '/'), we get the picture of Figure 8. From the second row labels $10, 6, 6, 2, 2$ and the fact that there are exactly 26 1-entries, we immediately obtain the following normalized Banzhaf indices of the parties:

$$\text{CIU: } \tfrac{10}{26} \quad \text{PSC-CPC: } \tfrac{6}{26} \quad \text{ERC: } \tfrac{6}{26} \quad \text{PP: } \tfrac{2}{26} \quad \text{ICV-EA: } \tfrac{2}{26}$$

If in these fractions the denominators 26 are changed to $2^{5-1} = 16$, then the results are the parties' absolute Banzhaf indices.

Next, we evaluate the expression Swingers(minwin($v$)). Then RelView depicts the labeled Boolean matrix of Figure 9 on its screen. Hence, the normalized Holler-Packel indices of the Catalonian parties are as follows:

$$\text{CIU: } \tfrac{3}{13} \quad \text{PSC-CPC: } \tfrac{3}{13} \quad \text{ERC: } \tfrac{3}{13} \quad \text{PP: } \tfrac{2}{13} \quad \text{ICV-EA: } \tfrac{2}{13}$$

In Example 4.2.1 we have shown that there are five minimal winning coalitions. As a consequence, a change of the denominators 13 to 5 yields the absolute Holler-Packel indices of the parties as (in the same order) $\frac{3}{5}$, $\frac{3}{5}$, $\frac{3}{5}$, $\frac{2}{5}$ and $\frac{2}{5}$. □

The Shapley-Shubik index, the Deegan-Packel index and the Johnston index are three further prominent power indices for measuring power in simple games. In contrast with the Banzhaf index and the Holler-Packel index, their definitions use more arithmetic operations than (2) and (3). As we will show in the next example by means of the Deegan-Packel index, in principle relation algebra and RelView can also be applied here. But the example also clearly shows the limit of the use of RelView in respect thereof.

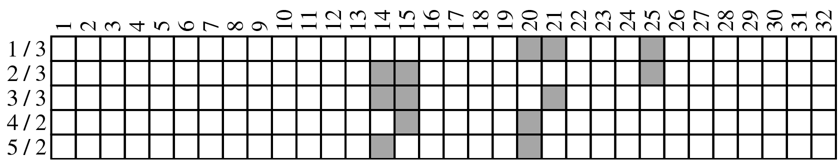


**Fig. 9.** The is-swinger relation wrt. the minimal winning coalitions

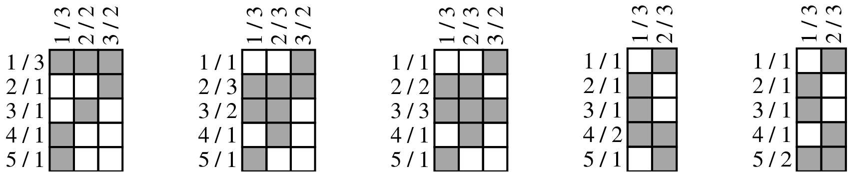

**Fig. 10.** Relations for determining the Deegan-Packel indices

**Example 4.4.2.** To compute the Deegan-Packel index $D(k)$ of a player $k \in N$ using relation algebra and the RelView tool, we assume the vector representation $m := \mathrm{minwin}(v)$ of the set $\mathcal{W}_{\min}$ of minimal winning coalitions to be at hand and the player $k \in N$ to be represented by a point $p : N \leftrightarrow \mathbf{1}$ in the sense of Section 3.2. If $\mathsf{E} : N \leftrightarrow 2^N$ is the is-element relation, then a little reflection shows that the vector $\mathsf{E}^\mathsf{T} p : 2^N \leftrightarrow \mathbf{1}$ represents the set of all coalitions $S \in 2^N$ such that $k \in S$ and, therefore, the relation

$$\mathrm{Deegan}(m,p) := \mathsf{E}\,\mathrm{inj}(m \cap \mathsf{E}^\mathsf{T} p)^\mathsf{T} \tag{10}$$

of type $[N \leftrightarrow \mathcal{W}^{(k)}_{\min}]$ column-wisely represents the set $\mathcal{W}^{(k)}_{\min}$ used in (4) to define $D(k)$. Based on $m$ and (10), now the Deegan-Packel index $D(k)$ can be determined by performing one after another the following three steps:

1. Compute for each column of $\mathrm{Deegan}(m,p)$ the reciprocal value of the number of its 1-entries.
2. Add all numbers obtained by the first step.
3. Divide the result of the second step by the number of 1-entries of $m$.

In the case of our Catalonian parliament example, the RelView tool delivered the five relations $\mathrm{Deegan}(m,p)$ which are depicted in Figure 10, where the point $p$ represents (from left to right) the five parties CIU, PSC-CPC, ERC, PP and ICV-EA. If we apply the above procedure, then we obtain from the second colum labels of these matrices that D(CIU) $= \frac{1}{5}(\frac{1}{3}+\frac{1}{2}+\frac{1}{2}) = \frac{8}{30}$, D(PSC-CPC) $= \frac{1}{5}(\frac{1}{3}+\frac{1}{3}+\frac{1}{2}) = \frac{7}{30}$, D(ERC) $= \frac{1}{5}(\frac{1}{3}+\frac{1}{3}+\frac{1}{2}) = \frac{7}{30}$, D(PP) $= \frac{1}{5}(\frac{1}{3}+\frac{1}{3}) = \frac{4}{30}$, and D(ICV-EA) $= \frac{1}{5}(\frac{1}{3}+\frac{1}{3}) = \frac{4}{30}$. □

It is obvious that the calculations of Example 4.4.2 hardly can be done by hand if the number of minimal winning coalitions is large. For instance, the situation becomes a good deal worse in the case of the present 10-parties Dutch parliament, since here already 42 of the 505 winning coalitions are minimal winning. To overcome the difficulties caused by the restrictive programming language of RelView,[4] in the course of the Ph.D. thesis Milanese (2003, [27]) and the M.Sc. thesis Szymanski (2003, [44]), the Kure library has been developed. It comprises the core functionality of RelView and opens the possibility to integrate relation-algebraic computations into C- and Java-programs. Particularly with regard to the above example, a use of Kure allows to perform all the arithmetic computations we have done by hand automatically by the superordinate C or Java-program.

[4] Caused by the specific application domain of the tool, relations are the only pre-defined datatype of this language and all further datatypes have to be modeled via them. In particular, real numbers and their base operations do not exist and it seems to be very difficult to model the reals in the same elegant and efficient way as, e.g., sets and a lot of structures of discrete mathematics and computer science.

## 5 Conclusion

We have presented two relation-algebraic models of simple games. For the vector model we have developed executable relation-algebraic specifications for testing certain fundamental properties of simple games and for computing specific players, specific coalitions and some relations between coalitions and between players and coalitions, respectively, which are important for determining dominance and power indices. The resulting algorithms have been executed by the BDD-based tool RelView after a straightforward translation into the tool's programming language for the example of the Catalonian parliament after the 2003 elections.

The algorithms we have used are expressed by extremely short and concise RelView-programs. They are easy to alter in case of slightly changed specifications. Combining this feature of the tool with its possibilities for visualization and stepwise execution of programs allows the user to experiment and play with new concepts while avoiding unnecessary expenditure of work. This makes the tool very useful for scientific research. Because of its visualization and stepwise execution facilities RelView is also most suitable for educational purposes. Another advantage of RelView is its implementation of relations via BDDs. Concerning efficiency it proved to be superior to many other well-known implementations, like Boolean matrices, lists of pairs and lists of successor/predecessor lists. This was especially of immense help for the problems we have treated in this paper. Many problems on simple games are hard since they require to compute or to check sets of exponential size and a lot of experiments have shown that precisely this is a strength of the tool. The reader is e.g., referred to Berghammer et al. (2002, [6]), Berghammer and Milanese (2006, [7]) and Berghammer and Fronk (2006, [4]), where further examples of the potential of RelView in this regard are presented. Due to the BDD-implementation, without any problems we have been able to apply our algorithms to a lot of simple games that originate from real political life and are e.g., presented in Peleg (1981, [31]), van Deemen (1989, [12]), and van Roozendaal (1990, [34]).

Of course, in spite of the fact that the system implements relations very efficiently, frequently RelView-programs cannot compete with special programs tailored for problems of the kind we have considered in Section 4 of the paper – although in the case of #P-complete or NP-hard problems the complexities are usually the same. It is known that a number of problems around simple games belong to these classes; see e.g., Prasad and Kelly (1990, [33]). We believe that the real attraction of RelView lies in its flexibility, its large application area, its computational power when dealing with enumerations of huge sets of 'interesting objects' (e.g., to verify an example or to construct a counter-example), its manifold animation and visualization possibilities, and the concise form of its programs. Nevertheless, it will be interesting to compare the competencies of the RelView tool with other game-theoretic algorithms such as the $O(n \cdot 2^{\frac{n}{2}})$ algorithm for the Banzhaf index and the $O(n^2 \cdot 2^{\frac{n}{2}})$ algorithm for the Shapley-Shubik index of weighted majority games presented in Klinz and Woeginger (2005, [22]) or the multilinear-extension-based algorithms mentioned in Alonso-Meijide et al. (2008, [1]) and Lorenzo-Freire et al. (2007, [26]).

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